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RAPIDLY CONVERGENT SERIES REPRESENTATIONS FOR $\zeta(2n+1)$ AND THEIR χ-ANALOGUE

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1. INTRODUCTION

Let $s=\sigma+it$ be a complex variable. The Riemann zeta-function $\zeta(s)$ is defined by

$$\zeta(s)=\sum_{m=1}^{\infty} m^{-s} \qquad (\operatorname{Re} s=\sigma>1),$$

and its meromorphic continuation over the whole s-plane, whose only singularity is a simple pole with the residue one.

For the specific values of $\zeta(s)$ at positive even integers, the formula

$$\zeta(2n)=(-1)^{n-1}\frac{(2\pi)^{2n}}{2(2n)!}B_{2n} \qquad (n=1,2,3,\ldots), \tag{1.1}$$

due to Euler, is classically known. Here B_n $(n\geq 0)$ is the Bernoulli number defined by the Taylor series expansion

$$\frac{z}{e^z-1}=\sum_{n=0}^{\infty}\frac{B_n}{n!}z^n \qquad (|z|<2\pi).$$

Closed form evaluations like (1.1), however, for the values of $\zeta(s)$ at positive odd integers have been unknown up to the present time.

It is the purpose of this paper to study rapidly convergent series representations for the values of $\zeta(s)$ at positive odd integers. We shall prove certain transformation formulae for

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Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX the power series including the values of $\zeta(s)$ at positive *even* integers in their coefficients (see Theorems 1 and 2 given below). A particular case of each of these formulae implies the previously known rapidly convergent series representations for the values of $\zeta(s)$ at positive odd integers. (One is classic, and the other is recently found.) A χ-analogue of our transformation formulae will also be given in Theorem 3.

It was found by Euler in 1772 (see Ayoub [Ay, p.1080, Section 7]) that $\zeta(3)$ has an infinite series representation

$$\zeta(3)=\frac{1}{7}\pi^2\left\{1-4\sum_{k=1}^{\infty}\frac{\zeta(2k)}{(2k+1)(2k+2)2^{2k}}\right\}. \tag{1.2}$$

This formula was rediscovered by Ramaswami [Ra] and (more recently) by Ewell [Ew1]. In fact, Euler's formula (1.2) was reproduced by Srivastava [Sr1, p.7, Equation (2.23)] from the work of Ramaswami [Ra]. Inspired by Ewell's rediscovery of (1.2), and his subsequent result [Ew2], Yue and Williams [YW] established a generalization of (1.2), which gives, though complicated, an exact series representation for $\zeta(2n+1)$ with any nonnegative integer n. The formula of Yue and Williams was considerably simplified by Cvijović and Klinowski [CK, Theorem A], who proved that

$$\begin{aligned}\zeta(2n+1)=(-1)^n\frac{(2\pi)^{2n}}{n(2^{2n+1}-1)}&\left\{\sum_{k=1}^{n-1}(-1)^{k-1}\frac{k\zeta(2k+1)}{(2n-2k)!\pi^{2k}}\right.\\&\left.+\sum_{k=0}^{\infty}\frac{(2k)!\zeta(2k)}{(2n+2k)!2^{2k}}\right\}\end{aligned} \tag{1.3}$$

for any positive integer n, where the finite sum on the right-hand side is to be regarded as null if $n=1$. Since $\zeta(0)=-1/2$, we see that (1.3) reduces to (1.2) when $n=1$.

Srivastava [Sr2] recently found the existence of certain families of rapidly convergent series representations for $\zeta(2n+1)$. Cvijović and Klinowski's formula (1.3) belongs to one of these families, while another family includes classical Wilton's [Wi] formula

$$\begin{aligned}\zeta(2n+1)=(-1)^{n-1}\pi^{2n}&\left\{\frac{1}{(2n+1)!}\left(\sum_{m=1}^{2n+1}\frac{1}{m}-\log\pi\right)\right.\\&\left.+\sum_{k=1}^{n-1}(-1)^k\frac{\zeta(2k+1)}{(2n-2k+1)!\pi^{2k}}+2\sum_{k=1}^{\infty}\frac{(2k-1)!\zeta(2k)}{(2n+2k+1)!2^{2k}}\right\}.\end{aligned} \tag{1.4}$$

From the observation of various series representations for $\zeta(2n+1)$ appearing in [Sr2], we may say that Cvijović and Klinowski's formula (1.3) is one of the formulae that have the simplest figure among these families. It is in fact possible to show that (1.3) is a particular case of a general transformation formula, that is

Theorem 1. *Let n be a positive integer, and x a real variable with $|x| \le 1$. Then we have*

$$\begin{aligned}(1.5)\qquad & n\zeta(2n+1) - n\sum_{l=1}^{\infty}\frac{\cos(2\pi lx)}{l^{2n+1}} - \pi x\sum_{l=1}^{\infty}\frac{\sin(2\pi lx)}{l^{2n}} \\ &= (-1)^n(2\pi x)^{2n}\Bigg\{\sum_{k=1}^{n-1}(-1)^{k-1}\frac{k\zeta(2k+1)}{(2n-2k)!(2\pi x)^{2k}} \\ &+\sum_{k=0}^{\infty}\frac{(2k)!\zeta(2k)}{(2n+2k)!}x^{2k}\Bigg\}.\end{aligned}$$

Remark. Since

$$\sum_{l=1}^{\infty}\frac{(-1)^l}{l^{2n+1}} = (2^{-2n}-1)\zeta(2n+1),$$

we see that the case $x = 1/2$ of Theorem 1 implies (1.3).

For the proof of Theorem 1 we treat the infinite sum on the right-hand side of (1.5), based on Mellin transform technique (see (2.1) and (2.2) below). This technique has the advantage of heuristic treatments, particularly for the infinite sums of the type mentioned above. Studies on certain power series and asymptotic series associated with the Riemann zeta and allied zeta-functions, based on this technique, were recently made by the author (see [Ka1] [Ka2] [Ka3]). The same technique also yields here another transformation formula, which includes Wilton's formula (1.4) as a particular case.

Theorem 2. *Let n be a positive integer, and x a real variable with $|x| \le 1$. Then we have*

$$\begin{aligned}(1.6)\qquad & \zeta(2n+1) + \frac{1}{2\pi x}\sum_{l=1}^{\infty}\frac{\sin(2\pi lx)}{l^{2n+2}} \\ &= (-1)^{n-1}(2\pi x)^{2n}\Bigg\{\frac{1}{(2n+1)!}\Bigg(\sum_{m=1}^{2n+1}\frac{1}{m} - \log 2\pi x\Bigg) \\ &+\sum_{k=1}^{n-1}(-1)^k\frac{\zeta(2k+1)}{(2n-2k+1)!(2\pi x)^{2k}} + 2\sum_{k=1}^{\infty}\frac{(2k-1)!\zeta(2k)}{(2n+2k+1)!}x^{2k}\Bigg\}.\end{aligned}$$

Remark. The formula which has a similar nature to (1.6) was proved in a quite different way by Ewell [Ew3, Throrem 1]. His formula yields a determinantial expression of $\zeta(2n+1)$, from which he derived exact infinite series representaions for $\zeta(2n+1)$ with $n = 1, 2$ and 3.

Furthermore, the proof of Theorem 1 suggests that a χ-analogue of (1.5) exists. Let q be a positive integer, χ a Dirichlet character of modulus q. We denote by $L(s,\chi)$ the Dirichet L-function attached to χ, and $\tau(\chi)$ Gauss' sum defined by

$$\tau(\chi) = \sum_{a=1}^{q}\chi(a)e^{2\pi ia/q}.$$

Theorem 3. *Let n be a positive integer, and x a real variable with $|x| \leq 1$. For any primitive character χ of modulus q, we have the following formulae.*

(i) If χ is an even character (i.e., $\chi(-1) = 1$),

$$\tag{1.7} \begin{aligned} &nL(2n+1,\chi) - n\sum_{l=1}^{\infty} \frac{\chi(l)\cos(2\pi lx/q)}{l^{2n+1}} - \pi x \sum_{l=1}^{\infty} \frac{\chi(l)\sin(2\pi lx/q)}{l^{2n}} \\ &= (-1)^n \left(\frac{2\pi x}{q}\right)^{2n} \left\{ \sum_{k=1}^{n-1} (-1)^{k-1} \frac{kL(2k+1,\chi)}{(2n-2k)!(2\pi x/q)^{2k}} \right. \\ &\quad \left. + \frac{\tau(\chi)}{q} \sum_{k=0}^{\infty} \frac{(2k)!L(2k,\overline{\chi})}{(2n+2k)!} x^{2k} \right\}; \end{aligned}$$

(ii) If χ is an odd character (i.e., $\chi(-1) = -1$),

$$\tag{1.8} \begin{aligned} &L(2n,\chi) - \sum_{l=1}^{\infty} \frac{\chi(l)\cos(2\pi lx/q)}{l^{2n}} \\ &= (-1)^n \left(\frac{2\pi x}{q}\right)^{2n-1} \left\{ \sum_{k=1}^{n-1} (-1)^{k} \frac{L(2k,\chi)}{(2n-2k)!(2\pi x/q)^{2k-1}} \right. \\ &\quad \left. + 2i\frac{\tau(\chi)}{q} \sum_{k=0}^{\infty} \frac{(2k)!L(2k+1,\overline{\chi})}{(2n+2k)!} x^{2k+1} \right\}. \end{aligned}$$

Remark. The shape of the left-hand side of (1.8) shows that this formula is rather a χ-analogue of (1.6).

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We shall prove Theorem 1 in the next section. Theorem 2 will be shown in Section 3. The last section will be devoted to the proof of Theorem 3.

2. Proof of Theorem 1

Let n be a fixed positive integer, x a real variable, and set

$$\tag{2.1} I(x) = \frac{1}{4i} \int_{(\sigma_0)} \cot\left(\frac{1}{2}\pi s\right) \zeta(s) \frac{x^s}{(s+1)(s+2)\cdots(s+2n)} ds \qquad (|x| \leq 1),$$

where σ_0 is a constant fixed with $-1/2 < \sigma_0 < 0$, and (σ_0) denotes the vertical straight line from $\sigma_0 - i\infty$ to $\sigma_0 + i\infty$. The integral in (2.1) converges absolutely, because the order of the integrand is bouded as $O(|t|^{\frac{1}{2}-\sigma_0-2n+\varepsilon})$, when $t \to \pm\infty$, with an arbitrary small $\varepsilon > 0$, by the vertical estimate $\zeta(s) = O(|t|^{\frac{1}{2}-\sigma+\varepsilon})$ for $\sigma < 0$ (cf. Titchmarsh [Ti, p.95, Chapter V]).

We start the proof of Theorem 1 with the observation that

$$I(x) = -\sum_{k=0}^{\infty} \frac{\zeta(2k)}{(2k+1)(2k+2)\cdots(2k+2n)} x^{2k} \qquad (|x| \le 1). \tag{2.2}$$

This can be shown by moving the path (σ_0) of the integral in (2.1) to the right, and collecting the residues of the poles at $s = 2k$ ($k = 0, 1, 2, \ldots$), because the order of the integrand is $O\{(K+|t|)^{-2n-1}|x|^K\}$, as $t \to \pm\infty$, on the line $\sigma = 2K+1$ ($K = 1, 2, \ldots$).

We next transform the integral in (2.1) by applying the functional equation

$$\zeta(1-s) = 2^{1-s}\pi^{-s}\cos\left(\frac{1}{2}\pi s\right)\Gamma(s)\zeta(s) \tag{2.3}$$

(cf. [Ti, p.16, Chapter II, (2.1.8)]), where $\Gamma(s)$ denotes the gamma function. Using (2.3) and the formula $\Gamma(s)\Gamma(1-s) = \pi/\sin \pi s$, we have

$$\cot\left(\frac{1}{2}\pi s\right)\zeta(s) = 2^s\pi^{s-1}\cos\left(\frac{1}{2}\pi s\right)\Gamma(1-s)\zeta(1-s). \tag{2.4}$$

Substituting this into the integral in (2.1) and changing the variable s into $1-s$, we obtain

$$I(x) = \frac{1}{2i}x\int_{(\sigma_1)} \sin\left(\frac{1}{2}\pi s\right)F(s)\zeta(s)(2\pi x)^{-s}ds, \tag{2.5}$$

where $\sigma_1 = 1 - \sigma_0$ and

$$F(s) = \frac{\Gamma(s)}{(s-2)(s-3)\cdots(s-2n-1)}.$$

Note that σ_1 satisfies $1 < \sigma_1 < 3/2$. Since $\zeta(s) = \sum_{l=1}^{\infty} l^{-s}$ converges absolutely for $\sigma = \sigma_1$, it follows from (2.5) that

$$I(x) = \frac{1}{2}\pi i x\sum_{l=1}^{\infty}\{f(2\pi i l x) - f(-2\pi i l x)\}, \tag{2.6}$$

where

$$f(z) = \frac{1}{2\pi i}\int_{(\sigma_1)} F(s)z^{-s}ds. \tag{2.7}$$

This integral converges absolutely for $|\arg z| \le \pi/2$, since the order of the integrand is $O\{|t|^{\sigma_1-\frac{1}{2}-2n}e^{-(\frac{1}{2}\pi-|\arg z|)|t|}\}$ as $t \to \pm\infty$ (cf. Ivic [Iv, p.492, Appendix, (A.34)]), and so that the interchange of the order of summation and integration is justified by the fact that $f(\pm 2\pi i l x) = O(l^{-\sigma_1})$ for $l = 1, 2, \ldots$. The identity

$$\frac{1}{(s-2)(s-3)\cdots(s-2n-1)} = \frac{1}{(s-1)\cdots(s-2n)} + \frac{2n}{(s-1)\cdots(s-2n-1)}$$

shows that

$$F(s) = \Gamma(s-2n) + 2n\Gamma(s-2n-1). \tag{2.8}$$

To evaluate the integral in (2.7), we need

Lemma. *Let σ_1 be a constant with $1 < \sigma_1 < 3/2$. For any positive integer $k \geq 2$ and any z with $|\arg z| \leq \pi/2$, we have*

$$\frac{1}{2\pi i}\int_{(\sigma_1)}\Gamma(s-k)z^{-s}ds = z^{-k}\left\{e^{-z} - \sum_{h=0}^{k-2}\frac{(-z)^h}{h!}\right\}. \tag{2.9}$$

Proof. Suppose first that $|\arg z| < \pi/2$. Then changing the variable s into $s+k$, we see that the left-hand side of (2.9) is equal to

$$\frac{1}{2\pi i}z^{-k}\int_{(\sigma_1-k)}\Gamma(s)z^{-s}ds.$$

We move the path (σ_1-k) of this integral to the left, with noting $1-k < \sigma_1-k < 3/2-k(< 2-k)$. Collecting the residues of the poles at $s=-h$ $(h = k-1, k, k+1, \ldots)$, we find that the left-hand side of (2.9) is further modified as $z^{-k}\sum_{h=k-1}^{\infty}(-z)^h/h!$. This proves Lemma for $|\arg z| < \pi/2$. The remaining case follows from the continuity of the integral in (2.9), since the order of the integrand is $O\{|t|^{\sigma_1-k-\frac{1}{2}}e^{-(\frac{1}{2}\pi-|\arg z|)|t|}\}$ for $|\arg z| \leq \pi/2$, when $t \to \pm\infty$. □

It follows from (2.7), (2.8) and Lemma that

$$\begin{aligned} &f(2\pi ilx) - f(-2\pi ilx) \\ &= -4n(2\pi ilx)^{-2n-1} + 4n(2\pi ilx)^{-2n-1}\cos(2\pi lx) \\ &\quad - 2i(2\pi ilx)^{-2n}\sin(2\pi lx) - 4\sum_{k=1}^{n-1}\frac{k}{(2n-2k)!}(2\pi ilx)^{-2k-1}. \end{aligned}$$

Substituting this into (2.6), we obtain

$$\begin{aligned} I(x) = &-n(2\pi ix)^{-2n}\zeta(2n+1) + n(2\pi ix)^{-2n}\sum_{l=1}^{\infty}\frac{\cos(2\pi lx)}{l^{2n+1}} \\ &+ \pi x(2\pi ix)^{-2n}\sum_{l=1}^{\infty}\frac{\sin(2\pi lx)}{l^{2n}} \\ &- \sum_{k=1}^{n-1}\frac{k\zeta(2k+1)}{(2n-2k)!}(2\pi ix)^{-2k}, \end{aligned}$$

which with (2.2) completes the poof of Theorem 1. □

3. Proof of Theorem 2

In this section we prove Theorem 2. The skeleton of the proof is the same as that of Theorem 1, so the details will be omitted. Throughout the following sections the constant σ_0 and σ_1 are fixed respectively with $-1/2 < \sigma_0 < 0$ and $1 < \sigma_1 < 3/2$.

We begin the proof with the integral

$$J(x) = \frac{1}{4i}\int_{(\sigma_0)} \cot\left(\frac{1}{2}\pi s\right)\zeta(s)\frac{x^s}{s(s+1)\cdots(s+2n+1)}ds \qquad (|x| \le 1). \tag{3.1}$$

Noting the facts $\zeta(0) = -1/2$, $\zeta'(0) = -(1/2)\log 2\pi$, and

$$\frac{1}{s(s+1)\cdots(s+2n+1)} = \frac{\Gamma(s)}{\Gamma(s+2n+2)} = \frac{s^{-1}}{\Gamma(2n+2)}\cdot\frac{1+\psi(1)s+O(s^2)}{1+\psi(2n+1)s+O(s^2)}$$

with $\psi(s) = (\Gamma'/\Gamma)(s)$, we see that the residue of the pole at $s = 0$ of the integrand in (3.1) is

$$-\frac{1}{2(2n+1)!}\left(\sum_{m=1}^{2n+1}\frac{1}{m} - \log 2\pi x\right).$$

Then moving the path of integration in (3.1) to the right, and collecting the residues of the poles at $s = 2k$ ($k = 0, 1, 2, \ldots$), we get

$$\begin{aligned} J(x) = &-\frac{1}{2(2n+1)!}\left(\sum_{m=1}^{2n+1}\frac{1}{m} - \log 2\pi x\right) \\ &-\sum_{k=1}^{\infty}\frac{\zeta(2k)}{(2k)(2k+1)\cdots(2k+2n+1)}x^{2k} \end{aligned} \tag{3.2}$$

On the other side, we substitute (2.4) into the integral in (3.1), then change the variable s into $1-s$, and obtain

$$J(x) = \frac{1}{2i}x\int_{(\sigma_1)} \sin\left(\frac{1}{2}\pi s\right)G(s)\zeta(s)(2\pi x)^{-s}ds, \tag{3.3}$$

where

$$G(s) = \frac{\Gamma(s)}{(s-1)(s-2)\cdots(s-2n-2)} = \Gamma(s-2n-2). \tag{3.4}$$

Remark. In comparison with (2.8), the gamma factor (3.4) does not split in this case; the evaluation of $J(x)$ becomes simpler than that of $I(x)$ in the preceding case.

Substituting the representation $\zeta(s) = \sum_{l=1}^{\infty} l^{-s}$ into the integral in (3.3) and changing the order of summation and integration, we obtain

$$J(x) = \frac{1}{2}\pi i x\sum_{l=1}^{\infty}\{g(2\pi i x) - g(-2\pi i x)\}, \tag{3.5}$$

where

$$g(z) = \frac{1}{2\pi i}\int_{(\sigma_1)} G(z)z^{-s}ds$$

for $|\arg z| \le \pi/2$. Hence by Lemma and (3.5),

$$J(x) = \frac{1}{2}(2\pi i x)^{-2n}\zeta(2n+1) + \pi x(2\pi i x)^{-2n-2}\sum_{l=1}^{\infty}\frac{\sin(2\pi l x)}{l^{2n+2}}$$
$$+\frac{1}{2}\sum_{k=1}^{n-1}\frac{\zeta(2k+1)}{(2n-2k+1)!}(2\pi i x)^{-2k}.$$

This with (3.2) establishes Theorem 2. □

4. Proof of Theorem 3

We first treat the even character case (i) of Theorem 3. In this case the functional equation is of the form

$$L(1-s,\overline{\chi}) = 2\tau(\chi)^{-1}\Big(\frac{2\pi}{q}\Big)^{-s}\cos\Big(\frac{1}{2}\pi s\Big)\Gamma(s)L(s,\chi) \tag{4.1}$$

(cf. Washington [Wa, p.29. Chapter 4]). This suggests to adopt the integral

$$K(x) = \frac{1}{4i}\int_{(\sigma_0)}\cot\Big(\frac{1}{2}\pi s\Big)L(s,\overline{\chi})\frac{x^s}{(s+1)(s+2)\cdots(s+2n)}ds \qquad (|x|\le 1), \tag{4.2}$$

as an initial setting. We first move the path (σ_0) to the right, passing over the poles at $s = 2k$ $(k = 0, 1, 2, \ldots)$ of the integrand, and obtain

$$K(x) = -\sum_{k=0}^{\infty}\frac{L(2k,\overline{\chi})}{(2k+1)(2k+2)\cdots(2k+2n)}x^{2k} \qquad (|x|\le 1). \tag{4.3}$$

Next changing the variable s into $1-s$ in (4.2), and then substituting (4.1), we get

$$K(x) = \frac{1}{2i}x\tau(\chi)^{-1}\int_{(\sigma_1)}\sin\Big(\frac{1}{2}\pi s\Big)F(s)L(s,\chi)\Big(\frac{2\pi x}{q}\Big)^{-s}ds,$$

and hence noting that $L(s,\chi) = \sum_{l=1}^{\infty}\chi(l)l^{-s}$ converges absolutely for $\sigma = \sigma_1$, we obtain

$$K(x) = \frac{1}{2}\pi i x\tau(\chi)^{-1}\sum_{l=1}^{\infty}\chi(l)\Big\{f\Big(\frac{2\pi i l x}{q}\Big) - f\Big(-\frac{2\pi i l x}{q}\Big)\Big\}.$$

Here $f(z)$ is given by (2.7). The evaluation of $f(2\pi i l x/q) - f(-2\pi i l x/q)$ is the same as in the proof of Theorem 1, so that

$$K(x) = -nq\tau(\chi)^{-1}\Big(\frac{2\pi i x}{q}\Big)^{-2n}L(2n+1,\chi)$$
$$+ nq\tau(\chi)^{-1}\Big(\frac{2\pi i x}{q}\Big)^{-2n}\sum_{l=1}^{\infty}\frac{\chi(l)\cos(2\pi l x/q)}{l^{2n+1}}$$
$$+ \pi x\tau(\chi)^{-1}\Big(\frac{2\pi i x}{q}\Big)^{-2n}\sum_{l=1}^{\infty}\frac{\chi(l)\sin(2\pi l x/q)}{l^{2n}}$$
$$- q\tau(\chi)^{-1}\sum_{k=1}^{n-1}\frac{kL(2k+1,\chi)}{(2n-2k)!}\Big(\frac{2\pi i x}{q}\Big)^{-2k}.$$

This with (4.3) establishes the former half of Theorem 3. □

We proceed to treat the odd character case (ii) of Theorem 3. The functional equation in this case asserts that

$$L(1-s,\overline{\chi}) = 2i\tau(\chi)^{-1}\Big(\frac{2\pi}{q}\Big)^{-s}\sin\Big(\frac{1}{2}\pi s\Big)\Gamma(s)L(s,\chi) \tag{4.4}$$

(cf. [Wa, p.29, Chapter 4]). This suggests to adopt the integral

$$H(x) = \frac{1}{4i}\int_{(\sigma_0)} \tan\Big(\frac{1}{2}\pi s\Big)L(s,\overline{\chi})\frac{x^s}{s(s+1)\cdots(s+2n-1)}ds \qquad (|x|\le 1), \tag{4.5}$$

as a starting point. We first move the path of integration in (4.5) to the right, passing over the poles at $s=2k+1$ $(k=0,1,2,\dots)$ of the integrand, and obtain

$$H(x) = \sum_{k=0}^{\infty}\frac{L(2k+1,\overline{\chi})}{(2k+1)(2k+2)\cdots(2k+2n)}x^{2k+1}. \tag{4.6}$$

Next changing the variable s into $1-s$ in (4.5) and using (4.4), we get

$$H(x) = -\frac{1}{2}x\tau(\chi)^{-1}\int_{(\sigma_1)}\cos\Big(\frac{1}{2}\pi s\Big)\Gamma(s-2n)L(s,\chi)\Big(\frac{2\pi x}{q}\Big)^{-s}ds.$$

This yields that

$$H(x) = \frac{1}{2}\pi i x\tau(\chi)^{-1}\sum_{l=1}^{\infty}\chi(l)\Big\{h\Big(\frac{2\pi i l x}{q}\Big)+h\Big(-\frac{2\pi i l x}{q}\Big)\Big\},$$

where

$$h(z) = \frac{1}{2\pi i}\int_{(\sigma_1)}\Gamma(s-2n)z^{-s}ds$$

for $|\arg z|\le \pi/2$. The evaluation of $h(2\pi i l x/q)+h(-2\pi i l x/q)$ is performed by Lemma, and it is seen that

$$\begin{aligned} H(x) = &-\frac{1}{2}q\tau(\chi)^{-1}\Big(\frac{2\pi i x}{q}\Big)^{-2n+1}L(2n,\chi)\\ &+\frac{1}{2}q\tau(\chi)^{-1}\Big(\frac{2\pi i x}{q}\Big)^{-2n+1}\sum_{l=1}^{\infty}\frac{\chi(l)\cos(2\pi l x/q)}{l^{2n}}\\ &-\frac{1}{2}q\tau(\chi)^{-1}\sum_{k=1}^{n-1}\frac{L(2k,\chi)}{(2n-2k)!}\Big(\frac{2\pi i x}{q}\Big)^{-2k+1}. \end{aligned}$$

This with (4.6) establishes the latter half of Theorem 3. The proof of Theorem 3 is therefore complete. □

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